# PhD project proposal: Problems in Extremal Combinatorics

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## 1 Introduction

Extremal combinatorics is a relatively young area of mathematics; as many students will know, it was founded in earnest by a group of mainly Hungarian mathematicians in the early 1900s, and was given huge impetus by Paul Erdős and his collaborators. There has been much exciting progress in extremal combinatorics in recent years, utilising techniques both from combinatorics itself, and also from other areas of mathematics such as algebra, analysis and probability theory. This PhD project would involve gaining familiarity with research-level techniques in combinatorics, and simultaneously tackling some unsolved problems in extremal combinatorics. Students would have a high degree of flexibility over what problems to work on.

Here is an example of a possible research problem. For any $d$, we let $Q_d$ denote the $d$-dimensional discrete hypercube graph, i.e., the graph with vertex-set $\{0,1\}^d$, where we join two zero-one vectors by an edge iff they differ in exactly one coordinate. Suppose we colour the edges of the discrete hypercube $Q_d$ with two colours. Two vertices of $Q_d$ are said to be *antipodal* if they are opposite one another, i.e., they are the maximum distance apart. Is it possible to find a pair of antipodal vertices which are joined by a path in which the colour of the edges changes only a small number of times? Feder and Subi [2] made the following conjecture.

**Conjecture 1.** *Let $d \in \mathbb{N}$ with $d \geqslant 2$. In any 2-colouring of $E(Q_d)$, there exists a pair of antipodal vertices and a path joining them along which the edge-colour changes at most once.*

This would be sharp, as can be seen by colouring all edges in one direction red and all the other edges blue.

Leader and Long [3] proved the following. (As usual, a geodesic in a graph is a path between two vertices of the minimum possible length.)

**Theorem 1.** *In any 2-colouring of $E(Q_d)$, there exists a monochromatic geodesic of length at least $\frac{d}{2}$.*

It follows from this theorem that there is a geodesic path joining two antipodal vertices with at most $\frac{d}{2}$ colour changes (just take any antipodal path extending a monochromatic geodesic of length $\lceil d/2 \rceil$). Another way of showing this is to note that the expectation of the number of colour changes on a random geodesic between a random antipodal pair of vertices is at most $\frac{d}{2}$. Dvŏrák [1] improved this by proving the following.

**Theorem 2.** *In any 2-colouring of $E(Q_d)$, there exists a pair of antipodal vertices and a path joining them along which the edge-colour changes at most $(\frac{3}{8} + o(1))d$ times.*

There is a big gap between Feder and Subi's conjecture and Dvŏrák's result. In particular, it is not known whether every 2-colouring of $E(Q_d)$ contains a pair of antipodal vertices and a path joining them along which the edge colour changes $o(d)$ times. Even proving this would be very interesting.

Here is another example of a possible research problem. A *nontrivial arithmetic progression of length three* is a set of the form $\{x, x+d, x+2d\}$, where $x, d \in \mathbb{R}$ with $d \neq 0$. We say a family $\mathcal{F}$ of subsets of $\{1, 2, \ldots, n\}$ is *3AP-intersecting* if for any $F, F' \in \mathcal{F}$, the intersection $F \cap F'$ contains a nontrivial arithmetic progression of length three. Simonovits and Sós conjectured that for any integer $n$, the largest possible size of a 3AP-intersecting family of subsets of $\{1, 2, \ldots, n\}$ is $2^{n-3}$: in other words, one cannot do better than to fix a single nontrivial arithmetic progression of length three and taking all subsets of $\{1, 2, \ldots, n\}$ containing that progression. This problem remains wide open. It is an example of an Erdős-Ko-Rado type problem, i.e., one concerning intersecting families of objects of a certain kind; there are many other interesting open problems in this area.

# References

[1] V. Dvořák. A Note on Norine's Antipodal-Colouring Conjecture. *Electronic J. Combin.* **27**, P2.26 (2020).

[2] T. Feder, C. Subi. On hypercube labellings and antipodal monochromatic paths. *Discrete Appl. Math.* **161**, 1421–1426 (2013).

[3] I. Leader and E. Long, Long geodesics in subgraphs of the hypercube. *Discrete Math.* **326**, 29–33 (2014).